Int. J. Heat Mass Transfer. Vol. 29, No. 10, pp. 1479–1485, 1986
Printed in Great Britain

0017–9310/86 $3.00 + 0.00
Pergamon Journals Ltd.

On the thermal adjustment of an almost-enclosed fluid region with through-flow

LARS RAHM
The Swedish Meteorological and Hydrological Institute, S-60176 Norrköping, Sweden

(*Received* 20 *June* 1985 *and in final form* 20 *February* 1986)

Abstract—The dynamics of a heat-up process in a horizontally enclosed fluid region, with an imposed through-flow and subject to weak thermal forcing, is investigated. Based on a boundary layer approach, a simple one-dimensional model for the interior fluid region is derived. The predicted response of the temperature field agrees well with numerical solutions of the corresponding undegenerate one-dimensional problem. The analytical results are in reasonably close agreement with numerical results presented in a companion paper by Hyun and Hyun (*Int. J. Heat Mass Transfer* **29**, 1487–1493 (1986)).

INTRODUCTION

A STUDY has been made of the thermal adjustment of a stably stratified fluid that is horizontally enclosed in a straight cylinder with a given through-flow. Cold fluid enters the vessel through the porous bottom, is heated at the vertical curved wall and leaves it via the porous top (see Fig. 1). (The geometry and boundary conditions have been chosen in order to simplify the analytical treatment). The fluid is assumed subject to an impulsive change in its boundary conditions. An investigation of the resulting thermal response is the purpose of this study. In fact, the problem of transient buoyant flows in a contained fluid has received little attention in the past, as remarked by Jischke and Doty [1] and later by Hyun [2].

The present work has partly been motivated by preliminary studies of planned commercial salmon farming on the Swedish west-coast. Deep-lying water is assumed pumped up into semi-enclosed containers so as to regulate the temperature of the contained water. This is done in order to improve the "environmental" conditions for the fish, which among other things control their growth. (In addition to this, a semi-enclosed container of the assumed type has also the advantage that the risk of unintentional eutrophication in a neighbourhood of the farming site can be minimized by simple means.) Another motivation for this study is the possibility of storing thermal energy in a single-phase fluid, contained in a simply-connected vessel, two reasons that make it economically attractive (cf. Gross [3]).

The present study is closely related to some earlier investigations by the author concerning both steady-state and transient stratified fluids. They are weakly thermally forced at their non-horizontal boundaries (Walin [5], Rahm and Walin [6, 7] and Rahm [8]). The study is based on the same theoretical background as those mentioned above (Walin [5]), but differs from them in that the interest is focused on the transient behaviour of the interior temperature field in an advection-dominated regime. This yields, to leading order, a degenerate interior dynamics where diffusion

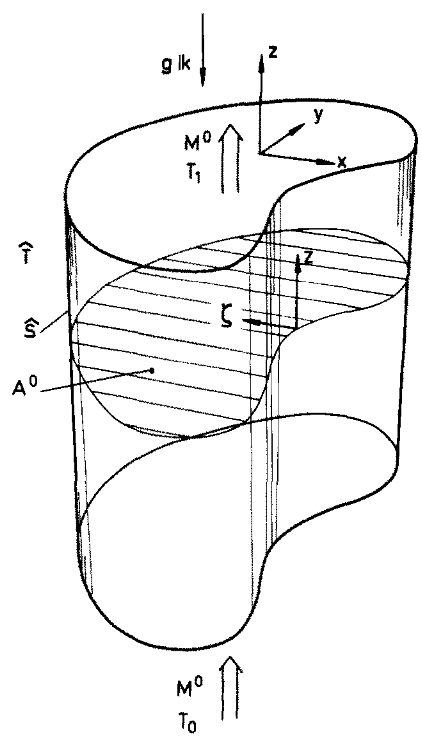

FIG. 1. Definition sketch showing the cylindrical vessel with the ambient temperatures T_0, $\hat{T}$ and T_1. The material property of the vertical boundary is expressed in the parameter $\hat{s}$. A flow M^0 is forced through the porous horizontal boundaries of cross-sectional area A^0 in the direction indicated by the arrows. The Cartesian coordinate system (x, y, z) as well as the boundary coordinate system (ζ, z) have their z-axis oppositely directed to the gravitational acceleration.

NOMENCLATURE

A^0	cross-sectional area
d	thickness of side wall
k	thermal conductivity
L	characteristic length-scale
M^B	buoyancy layer transport
M^0	through-flow
s	sidewall thermal conductance
t	time
T	temperature
W	vertical velocity
x, y, z	Cartesian coordinates.

Greek symbols

δ	bondary layer thickness
κ	thermal diffusivity
η	moving interior coordinate
τ	characteristic time
ζ, z	buoyancy layer coordinates
ξ, γ	stretched vertical coordinates.

processes are limited to only "interior" boundary layers. Analytical solutions have been derived for these time-dependent processes in a simple case with constant isotropic eddy diffusivity. (This is of course a severe idealization of the conditions prevailing in a container densely packed with salmon. Nevertheless, it may yield a qualitatively correct description of the "heat-up" process.) These results agree well with numerical solutions of the corresponding non-degenerate equations.

In a companion paper, Hyun and Hyun [4], numerical solutions to the Navier–Stokes equations for a Boussinesq fluid are presented for the same case as discussed in the present work. The numerical solutions yield flow and temperature details not readily obtainable from the analytical study. Further, the numerical results indicate the range of validity of the approximate boundary-layer solutions.

ANALYSIS

The theory [5] is valid under the following conditions: (i) The fluid is strongly stably stratified, yielding a horizontally homogeneous interior density field, and an interior dynamics, in which the momentum equations have degenerated into a simple hydrostatic balance. (ii) The strength of diffusion is limited, giving rise to the boundary layer character of the system. (iii) The heat flux through the vertical boundary is limited, enabling a linearization of the so-called "buoyancy layer". The interior heat equation then becomes to leading order

$$\frac{\partial T^{\mathrm{I}}}{\partial t}+W^{\mathrm{I}}\frac{\partial T^{\mathrm{I}}}{\partial z}=\kappa\frac{\partial^2 T^{\mathrm{I}}}{\partial z^2}, \tag{1}$$

where κ, T and W are the eddy diffusion coefficient of heat, the temperature and the vertical velocity respectively. The superscripts I and B denote the interior- and buoyancy-layer variables. The "interior" coordinate system used, (x, y, z), is shown in the definition sketch (Fig. 1), whereas t is the time variable. Let us assume a Newtonian heat flux condition at the curved vertical wall,

$$\frac{\partial T}{\partial \zeta}=\hat{s}(T-\hat{T}) \quad \text{at} \quad \zeta=0, \tag{2}$$

where the ζ-axis is directed along the inward normal from the boundary (see also Fig. 1). $\hat{T}$ is the ambient temperature, which in this case is chosen constant but may vary with height without invalidating the analysis. If the thermal conductivity of the wall and fluid are $\hat{k}$ and k, respectively, and the wall-thickness is d, then $\hat{s}$ is defined by

$$\hat{s}=\frac{\hat{k}}{kd}.$$

After integration around the vessel in a horizontal plane of cross-sectional area A^0 and utilizing boundary condition (2), the total buoyancy-layer transport becomes to lowest order (according to Ref. [5])

$$M^{\mathrm{B}}=-\kappa\oint\frac{\hat{s}(T^{\mathrm{I}}-\hat{T})}{\dfrac{\partial T^{\mathrm{I}}}{\partial z}}\,\mathrm{d}l. \tag{3}$$

The horizontally integrated continuity equation then becomes

$$M^{\mathrm{B}}+W^{\mathrm{I}}\cdot A^0=M^0.$$

Thus W^{I} can be expressed in terms of T^{I} and M^0 (the imposed through-flow). Equation (1) is then reformulated,

$$\frac{\partial T^{\mathrm{I}}}{\partial t}+\frac{M^0}{A^0}\frac{\partial T^{\mathrm{I}}}{\partial z}+\frac{\kappa}{A^0}\oint\hat{s}(T^{\mathrm{I}}-\hat{T})\,\mathrm{d}l=\kappa\frac{\partial^2 T^{\mathrm{I}}}{\partial z^2}, \tag{4}$$

which already satisfies condition (2). This equation is non-dimensionalized by the following transformation;

$$(x,y,z)=L(x',y',z')$$
$$(t)=\tau(t')$$
$$(T)=\Delta T(T')$$

where ΔT is also assumed valid for the external temperature field. Equation (4) then becomes (after dropping the primes)

$$\frac{\partial T^{\mathrm{I}}}{\partial t}+A\frac{\partial T^{\mathrm{I}}}{\partial z}+B(T^{\mathrm{I}}-\hat{T}) = C\frac{\partial^2 T^{\mathrm{I}}}{\partial z^2} \quad (5)$$

$$A = \frac{M^0\tau}{A^0 L} \quad (6a)$$

$$B = \frac{\kappa\tau\oint \hat{s}\,\mathrm{d}l}{A^0} \quad (6b)$$

$$C = \frac{\kappa\tau}{L^2}. \quad (6c)$$

A, B, and C are ratios of the characteristic time-scale τ to the imposed flush-time, the buoyancy-generated flush-time and the so-called diffusion time. As the interest is focused on the advection-dominated regime,

$$C \ll (A;B) \sim 1. \quad (7)$$

This yields to leading order

$$\frac{\partial T^{\mathrm{I}}}{\partial t}+A\frac{\partial T^{\mathrm{I}}}{\partial z}+B(T^{\mathrm{I}}-\hat{T}) = 0. \quad (8)$$

Obviously equation (8) cannot generally satisfy two boundary conditions simultaneously. Consequently the scaling assumptions done break down somewhere, indicating the existence of some type of boundary layer.

It turns out that the region containing the fluid that initially occupies the interior, with an initial temperature distribution

$$T^{\mathrm{I}} = \overset{\mathrm{Init}}{T}(z) \quad \text{at} \quad t = 0 \quad (9)$$

can be treated separately from the other region that is formed by the imposed flow M^0. This can be made use of by introducing a moving coordinate system

$$t^* = t$$

$$\eta^* = z - At.$$

It traverses upwards through the container with constant speed A in consort with M^0. The level $\eta^* = 0$ represents an interface separating the two interior regions. (The upstream and downstream regions are hereafter denoted by subscripts U and D respectively.) Equation (8) then becomes (drop the stars)

$$\frac{\partial T^{\mathrm{I}}}{\partial t}+B(T^{\mathrm{I}}-\hat{T}) = 0 \quad (10)$$

with a general solution, valid in both regions,

$$T^{\mathrm{I}} = \hat{T}+G(\eta)\exp\{-Bt\}. \quad (11)$$

The boundary conditions are (for simplicity)

$$T^{\mathrm{I}} = \begin{cases} T_1 & \text{at} \quad \eta = 1-At \quad (z=1) \quad (12a) \\ T_0 & \text{at} \quad \eta = -At \quad (z=0) \quad (12b) \end{cases}$$

Condition (12b) is obtainable either by a strong flow through a weakly heat-conducting porous bottom or vice versa. Condition (12a), however, is only obtainable by a weak through-flow through a strongly heat-conducting porous top lid.

The upstream region:

$$-At \leqslant \eta \leqslant 0; \quad 0 \leqslant t \leqslant A^{-1}$$

and

$$-At \leqslant \eta \leqslant 1-At; \quad t \geqslant A^{-1}.$$

The solution to equation (10), valid in the expanding upstream region, has to satisfy boundary condition (12b), which at $t = 0$ becomes its "initial condition",

$$T^{\mathrm{I}}_{\mathrm{U}} = \left[\hat{T}+(T_0-\hat{T})\exp\left(-\frac{Bz}{A}\right)\right]$$

$$= \hat{T}+(T_0-\hat{T})\exp\left[-\frac{B}{A}(\eta+At)\right]. \quad (13)$$

The downstream region:

$$0 \leqslant \eta \leqslant 1-At; \quad 0 \leqslant t \leqslant A^{-1}.$$

The solution to equation (10), valid in the contracting downstream region, must satisfy initial condition (9), thus

$$T^{\mathrm{I}}_{\mathrm{D}} = \hat{T}+[\overset{\mathrm{Init}}{T}(\eta)-\hat{T}]\exp(-Bt). \quad (14)$$

The interior

The complete interior solution is schematically shown in Fig. 2. The evolution of the temperature distribution in the downstream region is similar to the decaying stratification in a closed vessel subject to thermal forcing at its boundaries, which has been discussed in Rahm and Walin [7]. The temperature decays towards its ambient value at each level, within the moving frame of reference, irrespective of the stratification in adjacent regions. Though the amplitude of the initial temperature profile decreases in time, its "shape" is preserved. In a fixed frame of reference this fluid region is replaced by the upstream one. The interface, that separates them, acts like a "blind" in that it gradually "exposes" the new steady-state stratification, i.e. solution (13). (Note that (13) is time-independent in the Cartesian system.) This solution is equivalent to the lowest order steady state solution discussed (and experimentally verified) in Rahm and Walin [6] for an almost-enclosed fluid region with through-flow. However, the complete solution cannot satisfy the downstream boundary condition (12a). Further, there is a discontinuity in temperature at the interface. These features suggest the existence of "interior" boundary layers, both at the top and at the interface.

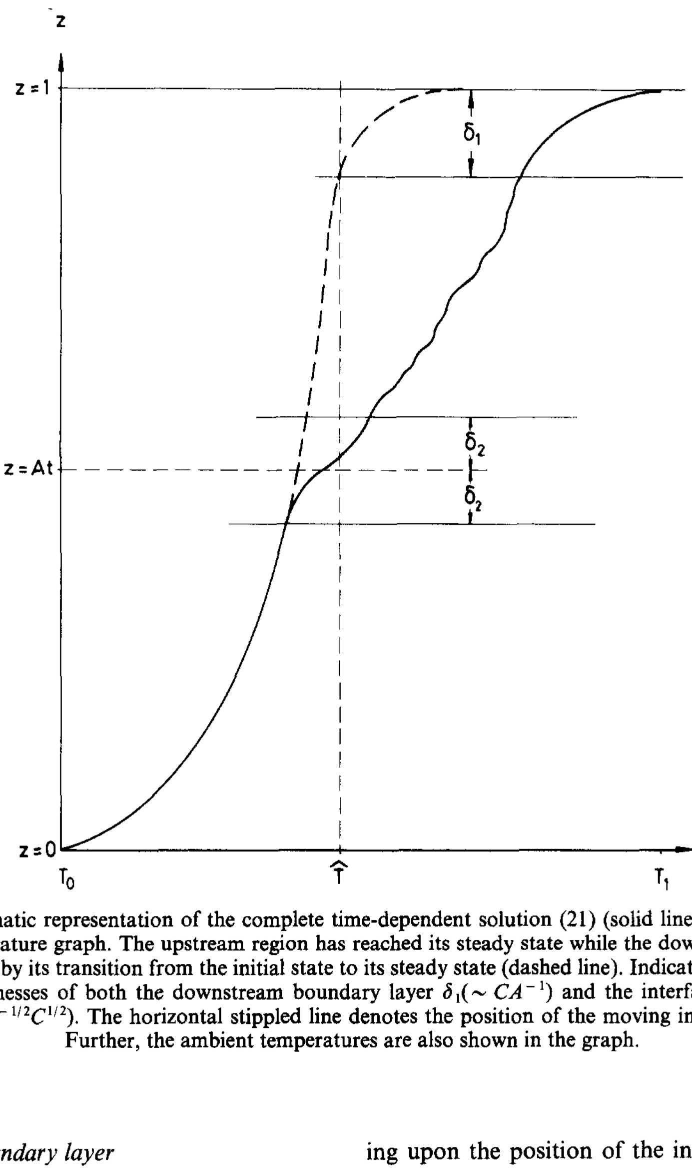

FIG. 2. Schematic representation of the complete time-dependent solution (21) (solid line) in this height versus temperature graph. The upstream region has reached its steady state while the downstream one is characterized by its transition from the initial state to its steady state (dashed line). Indicated in the figure are the thicknesses of both the downstream boundary layer $\delta_1(\sim CA^{-1})$ and the interfacial boundary layers $\delta_2(\sim B^{-1/2}C^{1/2})$. The horizontal stippled line denotes the position of the moving interface $z = At$. Further, the ambient temperatures are also shown in the graph.

The downstream boundary layer

A stretched variable (in the Cartesian system) is introduced, $1-z = \delta_1 \cdot \xi$. Simple scale analysis yields a boundary-layer thickness $\delta_1 \sim CA^{-1} = \kappa A^0(M^0L)^{-1}$, which is sufficiently thin to allow heat diffusion to balance the interior advective heat flux. The boundary layer equation becomes

$$-\frac{A}{\delta_1}\frac{\partial T^{\mathrm{DB}}}{\partial \xi} = \frac{C}{(\delta_1)^2}\frac{\partial^2 T^{\mathrm{DB}}}{\partial \xi^2}, \tag{15}$$

where the dependent downstream boundary-layer variables are denoted by superscript DB. Since the characteristic time-scale of the boundary-layer dynamics is much shorter than that of the interior process, it can match a slowly varying interior in a quasi-steady way. The general solution then becomes

$$T^{\mathrm{DB}} = H(t)\exp\left[\frac{A}{C}(\eta + At - 1)\right], \tag{16}$$

which will satisfy the boundary condition (12a) together with the interior solutions (13) or (14), depending upon the position of the interface. The solution turns out as

$$T^{\mathrm{DB}} = [T_1 - T^{\mathrm{I}}_{\mathrm{U,D}}(1-At, t)]\exp\left[\frac{A}{C}(\eta + At - 1)\right], \tag{17}$$

where $T^{\mathrm{I}}_{\mathrm{U,D}}$ represents one of the two interior solutions which is evaluated at the top. Note that this type of boundary layer can only exist at a downstream boundary.

The interfacial boundary layers

The discontinuity at the interface in the lowest-order temperature field is a consequence of the non-dissipative interior dynamics. It necessitates "intermediate" boundary layers (denoted by IB) that match the two interior solutions. By stretching η, $\eta = \delta_2 \cdot \gamma$, the boundary layer equation becomes

$$\frac{\partial T^{\mathrm{IB}}}{\partial t} + BT^{\mathrm{IB}} = \frac{C}{(\delta_2)^2}\frac{\partial^2 T^{\mathrm{IB}}}{\partial \gamma^2}, \tag{18}$$

where $\delta_2 \sim C^{1/2} \cdot B^{-1/2}$ is the fully developed boundary-layer thickness, which is reached on a time-scale equivalent to that of the interior process itself. There are two solutions, one for each interior region. Continuity requirements in both the temperature and its gradient yield

$$T^{IB}_{U,D} = 0 \qquad \text{at } t = 0 \quad (19a)$$

$$T^{I}_{U} + T^{IB}_{U} = T^{I}_{D} + T^{IB}_{D} \qquad \text{at } \eta = 0 \quad (19b)$$

$$\frac{\partial}{\partial \eta}(T^{I}_{U} + T^{IB}_{U}) = \frac{\partial}{\partial \eta}(T^{I}_{D} + T^{IB}_{D}) \quad \text{at } \eta = 0. \quad (19c)$$

The complications that arise, as the interface is close to one of the physical boundaries, are ignored. The solution is readily obtained by Laplace-transformation;

$$T^{I}_{U,D}(\eta, t) = D_1\left(\frac{t}{\pi}\right)\exp\left(\frac{-\eta^2}{4Ct} - Bt\right)$$
$$\pm\left[D_2 - \frac{D_1|\eta|}{(4Ct)^{1/2}}\right]\exp(-Bt) \quad (20a)$$

$$D_1 = \left[\frac{\partial \overset{\text{Init}}{T}(0)}{\partial \eta} + \frac{B}{A}(T_0 - \hat{T})\right]C^{1/2} \quad (20b)$$

$$D_2 = \left[\frac{\overset{\text{Init}}{T}(0) - T_0}{2}\right]. \quad (20c)$$

RESULTS

The complete "interior" solution is composed of its various elements;

$$T^{I} = \begin{cases} T^{I}_{U} + T^{IB}_{U} + T^{IB}_{D} + T^{I}_{D} + T^{DB} & \text{for } 0 < t < A^{-1} \quad (21a) \\ T^{I}_{U} + T^{DB} & \text{for } t > A^{-1}. \quad (21b) \end{cases}$$

It is valid in the entire interior region except when the interface is close to one of its physical boundaries (for an illustration of the general solution, see Fig. 2). From a mechanistic point of view, the heat-up process can be described as follows. The cold fluid that is pumped into the vessel through the bottom cannot initially pass the vessel via the buoyancy layer as the buoyancy-layer transport M^{B} is always determined, in the downstream region, by the interior stratification and the ambient temperature distribution. It must therefore take the route via the interior and consequently it will "lift" the downstream region. Hence it determines the physical location of both the downstream and the upstream regions. In the upstream region the interior vertical velocity is weak and directed downwards. Hence the imposed flow takes the route through the vessel via the buoyancy layer in the upstream region, whereafter it enters the interior at the level of the interface. The decay towards the ambient temperature occurs on a typical time-scale B^{-1} for both regions while the motion of the interface through the container has its own time scale, A^{-1}. As a conclusion, the heat-up process is rather fast in the present parameter range.

The inviscid interior has, however, discontinuities in the temperature field both at the top and at the interface separating the two interior regions. This degeneracy leads to the previously mentioned "interior" boundary-layer formation. The two discussed types of boundary layers are dynamically very different, something that is also reflected in their typical time scales. The interfacial boundary layers have a time-scale equal to the interior process itself, B^{-1}, while the downstream boundary layer has an adjustment time of CA^{-2}. (The latter layer is also essential in the steady-state solution.) Finally note that the internal-gravity oscillations of buoyancy frequency are filtered out in this model as is discussed by Hyun [2] in a paper dealing with a somewhat similar problem but for a completely closed vessel. It will not effect the lowest-order temperature field in a significant way but the velocity field is severely influenced.

The behaviour of the system is illustrated by an example, see Fig. 3. The analytical solution (21), is compared with the numerical solution to the associated undegenerate equation (5). The latter equation, in its finite difference form, was integrated forward in time using an explicit scheme. A linear temperature profile was chosen as an initial temperature distribution. The evolution of the numerical solution is illustrated by the instantaneous temperature profile for five different times. The numerical "solutions" collapse completely on the corresponding profiles obtained from the analytical solution. Even for moderately large values of the "expansion" parameter C, the agreement found is excellent, as is evident in Fig. 4. The analytical solution is slightly "warmer" than its numerical counterpart, as may be expected due to the degenerate (non-dissipative) governing equation (8).

However, the numerical experiments presented in Ref. [4] concerning the same problem show both a good agreement with the results of the degenerate model and the limitations inherent to the theory for both weak and strong thermal forcing. The limitations are attributed to endwall effects and the linearization of the buoyancy-layer equations (cf. Hyun and Hyun [4]).

Acknowledgements—I want to express my gratitude to the staff of the Department of Oceanography, University of Gothenburg, for their help in preparing the manuscript. The work herein reported was done while I had a fellowship from the Nordic University Group on Physical Oceanography.

REFERENCES

1. M. C. Jischke and R. T. Doty, Linearized buoyant motion in a closed container. *J. Fluid Mech.* **71**, 729–754 (1975).

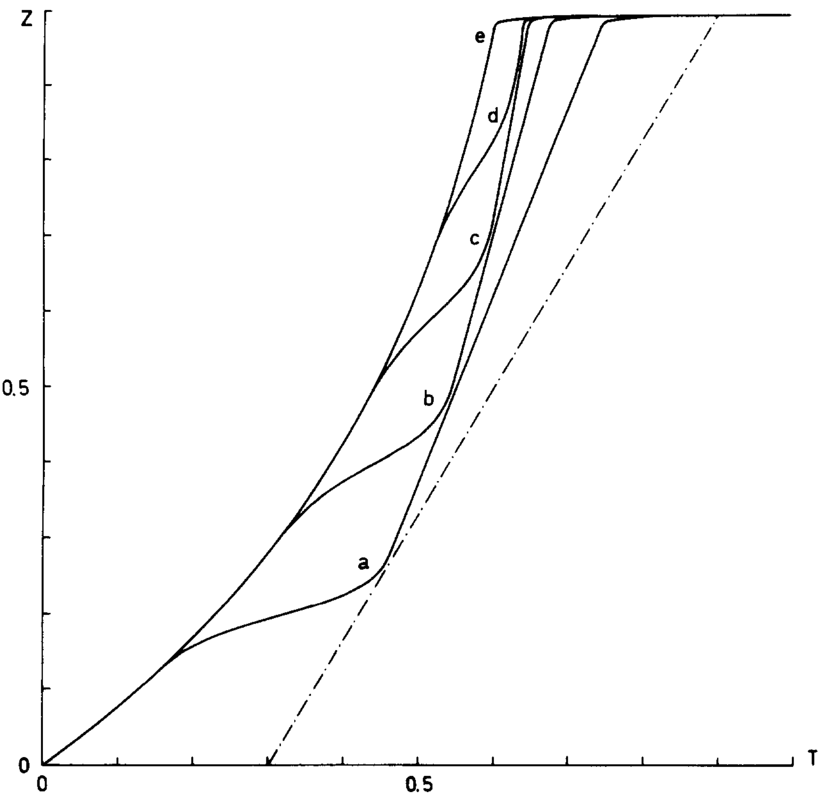

FIG. 3. A plot of the complete analytical solution (21) for $A = 1$, $B = 2$ and $C = 0.002$ at time $t = 0.2$ (a), $t = 0.4$ (b), $t = 0.6$ (c), $t = 0.8$ (d) that satisfies the boundary conditions $T_0 = 0$ and $T_1 = 1$. These results collapse completely on the numerically computed ones. The initial state $\overset{\text{Init}}{T}(z) = 0.3+0.6z$ is indicated by the stippled line. The steady-state solution is indicated by (e).

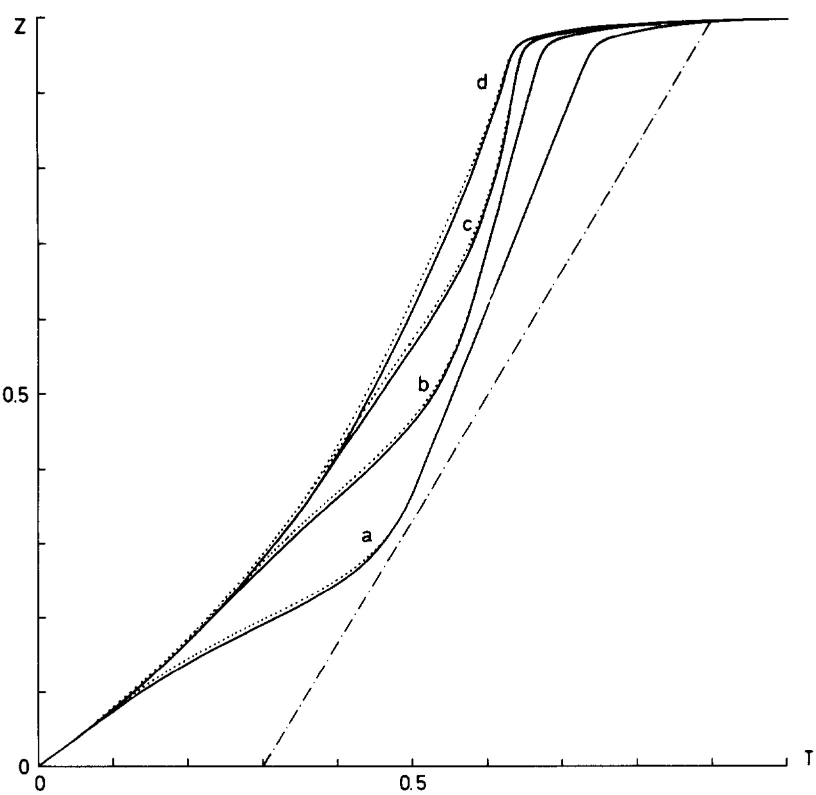

FIG. 4. Same legend as in Fig. 3 except that $C = 0.02$. The corresponding numerically computed solutions are also shown in the figure (the dotted lines).

2. J. M. Hyun, Transient buoyant convection of a contained fluid driven by the changes in the boundary temperatures. *ASME J. Appl. Mech.* **52**, 193–198 (1984).
3. R. J. Gross, An experimental study of single medium thermocline thermal energy storage. ASME Report 82-HT-53 (1982).
4. J. M. Hyun and J. C. Hyun, Adjustment of a thermally-stratified fluid in a container with vertical through-flow. Accepted for publication in *Int. J. Heat Mass Transfer* **29**, 1487–1493 (1986).
5. G. Walin, Contained non-homogeneous flow under gravity or how to stratify a fluid in the laboratory. *J. Fluid Mech.* **48**, 647–672 (1971).
6. L. Rahm and G. Walin, Theory and experiment on the control of stratification in almost-enclosed containers. *J. Fluid Mech.* **90**, 315–325 (1979).
7. L. Rahm and G. Walin, On thermal convection in stratified fluids. *Geophys. Astrophys. Fluid Dynamics* **13**, 51–65 (1979).
8. L. Rahm, A note on the heat up of an initially isothermal fluid. *Int. J. Math. Mod.* **6**, 19–30 (1985).

SUR L'AJUSTEMENT THERMIQUE D'UNE REGION FLUIDE PRESQUE CONFINE AVEC UN ECOULEMENT DE TRAVERSEE

Résumé—On étudie la dynamique du mécanisme de réchauffement dans une région de fluide confinée horizontalement, avec un écoulement imposé de traversée soumis à un faible forcement thermique. Basé sur une approximation de couche limite, un modèle simple monodimensionnel est conçu pour la région intérieure de fluide. La réponse calculée du champ de température s'accorde bien avec les solutions numériques du problème correspondant monodimensionnel non dégénéré. Les résultats théoriques sont en accord raisonnable avec des résultats théoriques présentés dans un article d'accompagnement de Hyun et Hyun [4].

DIE AUSBILDUNG DES TEMPERATURFELDES IN EINEM FAST EINGESCHLOSSENEN FLUIDGEBIET MIT DURCHSTRÖMUNG

Zusammenfassung—Die Dynamik des Aufheizvorgangs in einem horizontal eingegrenzten Fluidgebiet mit erzwungener Durchströmung und schwacher Beheizung wird untersucht. Beruhend auf einer Grenzschichtnäherung wird ein einfaches eindimensionales Modell für den inneren Fluidbereich abgeleitet. Die vorhergesagte Reaktion des Temperaturfeldes stimmt gut mit numerischen Lösungen des entsprechenden gewöhnlichen eindimensionalen Problems überein. Die analytischen Ergebnisse stimmen ziemlich gut mit den numerischen Ergebnissen aus der Veröffentlichung von Hyun und Hyun überein [4].

О ТЕРМОРЕГУЛЯЦИИ ЧАСТИЧНО ЗАКРЫТОГО ОБЪЕМА ЖИДКОСТИ СО СКВОЗНЫМ ПОТОКОМ

Аннотация—Изучается динамика процессов прогрева в горизонтально ограниченном объеме жидкости с наложенным сквозным потоком при слабом подводе тепла. В приближении пограничного слоя получена простая одномерная модель для внутреннего объема жидкости. Рассчитанное распределение температурного поля хорошо согласуется с численным решением невырожденной одномерной задачи. Аналитические результаты достаточно хорошо соответствуют численным данным работы [4].